Counterexamples in Mathematics Education: Why, Where, and How? – Software aspect

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Abstract
A counterexample is an example that refutes the propriety of some statement. For a mathematician, constructing counterexamples is a common way to disproof mathematical conjectures. Counterexamples also help her to establish the constraints imposed on theorems.
This paper shows that in mathematics education, counterexamples can and should be applied at the earliest stages - in the study of concepts, long before the first acquaintance with the theorems and proofs. Herewith, the use of software becomes an organic element of the learning process.

Studies of concepts

This is the first article in the series about counterexamples. We consider here the place and role of counterexamples in teaching of concepts with the most frequent logical structure of the propositional definition.

1. Propositional definitions

1.1. The principle of variation of non-essential features. From the very early moments of their life, children meet mathematical objects and acquire skills of their recognition. Pedagogical psychology provides the *principle of variation of non-essential features* [1] (while the essential features are kept invariable) which intended:

> To illustrate essential features of a concept by demonstrating various visual materials and instances, or to highlight essential characteristics of a concept by varying non-essential features. The goal of using variation is to help students understand the essential features of a concept by differentiating them from nonessential features and further develop a scientific concept [2].
>
> The common, which in this approach is identified with the essential, repeatedly acting on the person, each time reinforces the reflection of this common. Non-essential, on the contrary, every time in a new concrete form, is erased, does not enter into the content of the concept.
>
> In this case, non-essential features should not only vary, but also be contrasted with essential ones. That is, simultaneously with the consideration of objects in which non-essential characteristics vary, the teacher must emphasize that essential characteristics remain unchanged.

A widespread example in use of this principle in mathematics teachers' literature consists in recommendation to illustrate the notion of right triangle by drawing it in different orientations, not exclusively with vertical and horizontal catheti. One can add variations of color, material, size etc. if she fears, that in case of all models at the lesson made from blue cardboard, students will think, that in order to be a right triangle the figure should: a) be a triangle, b) have a right angle, c) have a blue color, and d) made from cardboard.

In this situation, the teacher faces a problem: which of the non-essential attributes should be varied and which could be omitted? In fact, unlike the finite number of essential features there are "infinitely" many non-essential ones.

Needless to say that the same situation exists in the development of software intended to facilitate the formation and assimilation of concepts.

...It is impossible to implement such an unconstructive pedagogical approach.

This principle has been convincingly criticized in psychological literature [3]. Without denying its role in situations of uncontrolled or poorly managed assimilation of concepts, it expresses the associative approach in understanding of the process of assimilation of concepts.

1.2. An active recognition of belonging to the concept. We will treat this principle as a secondary way of learning, and consider instead

an active recognition, based on the definition of mathematical concept.

In fact, each definition consists of *contextual* and *logical* constituents. While the contextual one (features of the concept) is specific, the logical structure of definition is extremely stable and very often coincides with the definitions of various notions. Thus, the majority of concepts studied in elementary, secondary, and high school have a conjunctive definition with two or three attributes.

There are 2^n combination of values in the truth table, which describes the logic of definition with n attributes. Clearly, it will be sufficient to consider only these cases in the concept's studies. Each case presents a ***type of task*** *- objects with definite combination of truth values of the essential-only features*.

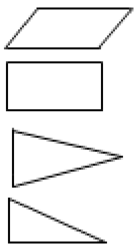
Any conceivable recognition object necessarily represents and only one of these types.

Figure 1 shows truth tables, modeling the types of tasks with proper examples-objects devoted to the concepts' recognition. The illustrated cases of two concepts - *right angle* and *monotonic function (in* **R***)* - with conjunctive and disjunctive definition help to grasp the following general conclusions:

- **Counterexamples are essential types of tasks to be studied**. From being an exotic educational tool, they become a vital learning element.
- There is a place to use the principle of non-essential features variation to provide numerous objects, representing the same type of task.
- There is a clear mechanism of task construction, which, being based on the combination of logical values, takes into account the concrete contents of the studied concept's features.

The selection of tasks no longer depends on the teacher's tastes. In case of difficulties in recognizing the object, students are offered another object of the same type in a pedagogically justified approach.

1	2	$\in, \notin$
f	f	f
f	t	f
t	f	f
t	t	t



Right triangle:
1. Triangle

AND

2. Has a right angle

1	2	$\in, \notin$	
f	f	f	$y = x^2$
f	t	t	$y = -x$
t	f	t	$y = x$
t	t	t	$y = 5$

Monotonic function:
1. Nondecreasing function

OR

2. Nonincreasing function

Figure 1

We got a constructive way of software development with a very limited amount of concrete types to support. Fortunately, these types are universal and cover a wide spectrum of studies of mathematical concepts.

1.3. Algorithm of recognition. The *activity approach to formation of concepts* [3], [4] mainly assumes organization of a controlled process *of recognizing the belonging of an object to the studied concept.* Naturally, this process is established on *an algorithm, based on the logic of the definition*. Instead of guessing whether the object belongs to concept (represents it) or not, students make their conclusion on the basis of a coherent check of the presence of attributes.

Surprisingly, *following this approach we obtain an additional type savings*, presented in Figure 2 by expressions: instead of 2^n types in case of *n* attributes one can use only *n*+1.

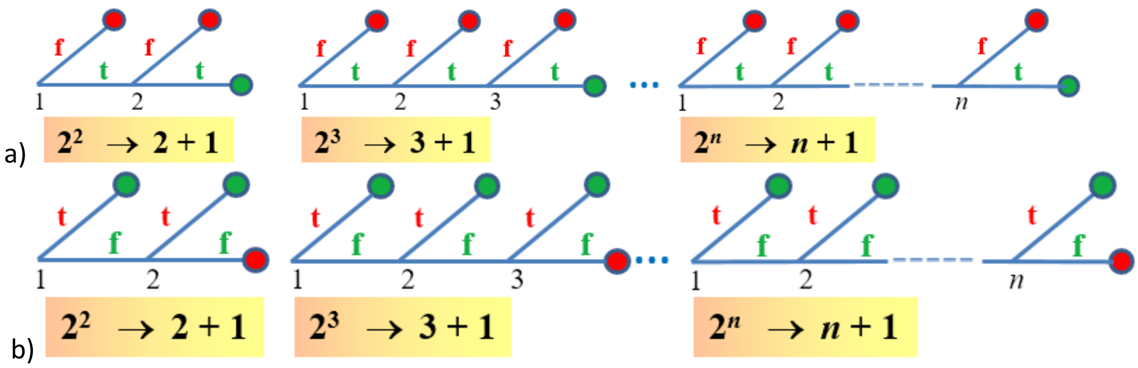


Figure 2: a) conjunctive definition, b) disjunctive definition.
Green dots - examples, Red dots - counterexamples.

For instance, having discovered the absence of the first characteristic ("triangle") in the counterexample-upper figure in Figure 1, one can stop checking the presence of the rest and conclude that this figure is not a right-angled triangle (see the leftmost red node of the recognition tree in Figure 2 a).

Motivation of such an early conclusion draws the students' attention to the specifics of the logical structure of the definition, and stimulates their logical thinking.

1.4. Ternary logic of recognition. Human logic is not binary. We have a wide "**Unknown**" between "Yes" and "No". Logic becomes ternary with *true, unknown,* and *false* possible values of truth. Figure 3 shows the formal truth tables and recognition algorithms[1]. As before, each dot presents a definite type of task-object of recognition.

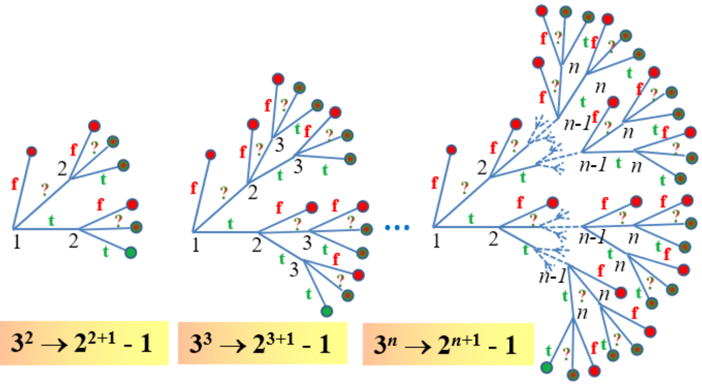
Surprisingly, ***the vast majority of tasks are counterexamples***.

Figure 4 shows counterexamples-objects with value of "unknown" (blurs) for some of their features:

a) The negative result of checking the first feature ("triangle") in Figure 4 a) immediately leads to the concrete conclusion, that the object is not a right-angled triangle, although it is impossible to find out whether it has a right angle or not. "It just does not matter". This activity models the path to very left red dot of the first graph in Figure 3 a).

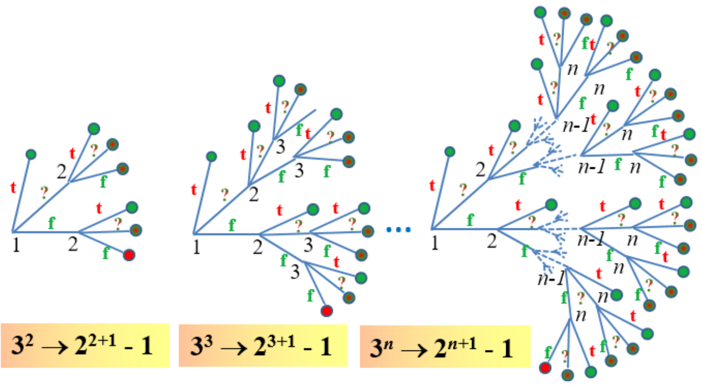
b) It is unknown whether Figure 4 b) shows the graph of a *monotonic function* or not (all depends on its behavior under the blur). So, the first feature gets value *unknown* ("?"). But we should proceed and check the second feature: *nonincreasing function*. Maybe our object has it, and hence we deal with a monotonic function. Unfortunately the answer is *false*. Conclusion: it is *unknown* – whether an object is a monotone function or not. This activity models the firth path (central) green-red node of the left graph in Figure 3 b).

1	2	∈, ∉
f	f	f
f	?	f
f	t	f
?	f	f
?	?	?
?	t	?
t	f	f
t	?	?
t	t	t



a)

1	2	∈, ∉
f	f	f
f	?	?
f	t	t
?	f	?
?	?	?
?	t	t
t	f	t
t	?	t
t	t	t



b)

Figure 3: a) con*junctive definition, b) disjunctive definition.*
Green dots - *examples,* Red & green-red ("*unknown*") dots - *counterexamples*.
Type savings: instead of 3^n types in case of n attributes one can use only 2^{n+1}-1 type.

[1] Of course, these two logical structures do not exhaust all possibilities, but they represent the overwhelming majority.

a) b)

Figure 4. Counterexamples-objects with value of “unknown” of attribute

Obviously, it is impossible to create a single software that would generate concrete objects (examples and counterexamples) representing these types of tasks - they are specific to each concept in accordance with its content, but this can be done ***concerning the logical component of the definition.***

1.5. Implementation in the software. The presented approach is implemented in the author’s Microsoft Excel application (Figure 5). It includes:

- Text boxes for entering the textual description of features (1) and term (2) of the studied concept.
- Arrow-shaped buttons (3) for selection of the proper logical structure[2].
- Button (4) to increase/decrease amount of features.
- Feature truth input buttons (5): green–YES, yellow–UNKNOWN, red–NO.
- Button (6) for automatic conclusion.
- Button (7) to hide/show the group of three input colored buttons.
- Button (8) to show/hide task-object.
- Radio buttons of binary/ternary logical modeling choice (9).
- Check button of objects types’ random generation (10).
- Reset button (11).

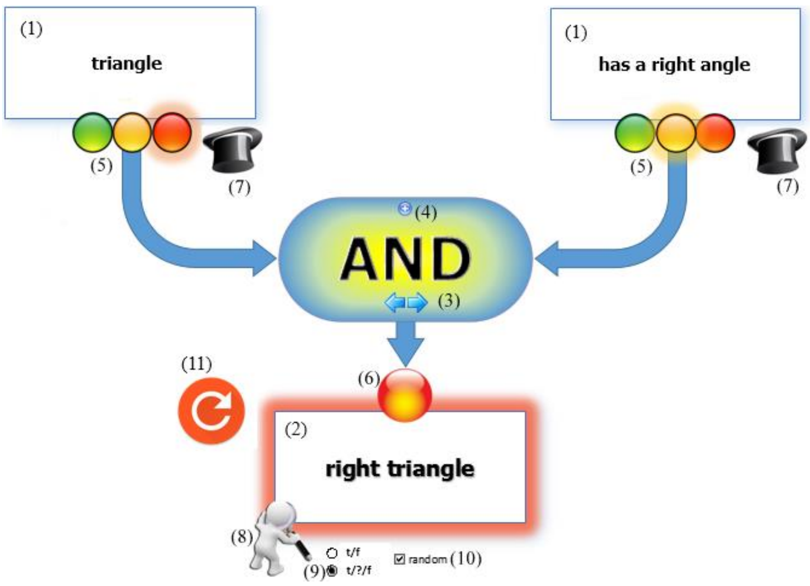


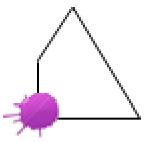
Figure 5. General view of application

[2] The possible option is: AND, OR, NOT, ⇔, ⇒. The last two options are especially useful while studying theorems.

The following fragment of printed instructional material presents one of the possible ways of this tool application:

General instructions:

1. Consider definition of the concept.
   Identify
   - Term,
   - Properties,
   - Type of properties' link (logical operator: AND, OR).
2. Select logical operator (AND, OR) by means of arrows (3), and amount of properties by button (4).
3. Input proper features and term descriptions in text boxes (1) and (2).
4. Disable automatic conclusion mechanism - press button (6) to make the button blue and to remove the colored halo around the term's text box.
5. Set all triples of input buttons to the “unknown” state – press “*reset*” button (11).
   …

Task #...:[3] Is the figure  a right triangle?

Solution:

- Check the first property (*Is it a* ____________?).
- It__________. So, press the ______________ button.
  passes, fails *green, yellow, red*
- *Is it sufficient to conclude whether the object is an example of right angle or not*? _____.
  yes, no

If your answer is “*yes*”, check it as follows:

a. Press the button of automatic answering (6). -It becomes red.

b. Try pressing each one of three unput buttons below the text of second feature and observe the color around the text of term.

- *Is the color remains the same*? _____.
  yes, no
- *What does it mean: the same or different colors in terms of the need to continue checking*?____.
- Actually, the color remains _______________.
  green, yellow, red

So, we stop and conclude that the object is ___________ a *right triangle*.
not, definitely

[3] We give here only one example. In fact, students are offered several exercises of each type. In the first of them hints are maximal and texts are very concrete: "Is there a first property?" etc. In the following examples of the solution, the terminology becomes more general: "Is the figure a right triangle?" The prompt level becomes minimal.

The **system of tasks types**, which examples presented in Figure 3, **is optimal:** ***necessary and sufficient.***[4] In fact, the absence of study of the path guiding to some dot leads to knowledge gaps. On the other hand, all possible logical routes of the recognition process are modeled by the corresponding tree.

The *construction of a universal mechanism for automatic generation of objects is impossible* due to the specific nature of their content. For each concept such a mechanism should be created separately. The Microsoft Excel application considered includes such a mechanism, supporting studies of certain concepts. Figure 6 shows the interface window in studies of "*linear pair of angles*.[5]

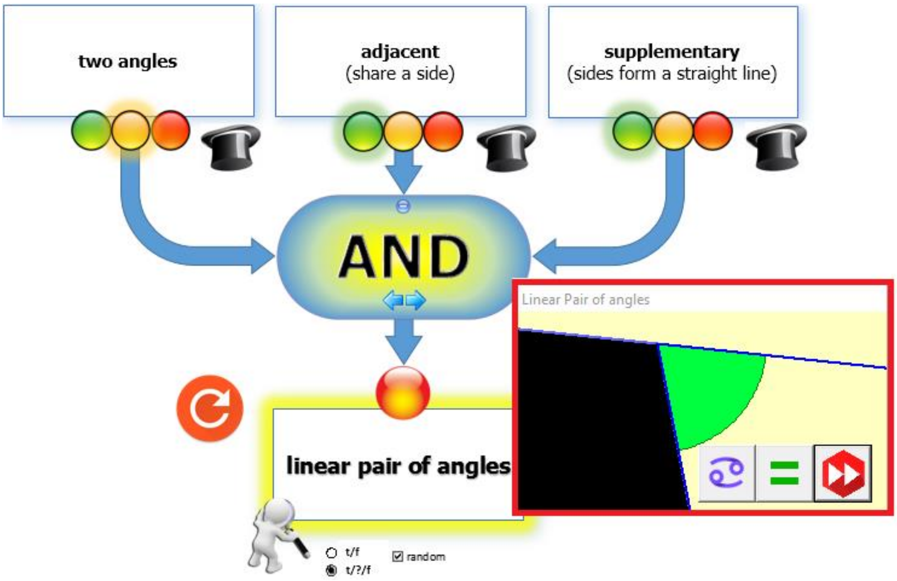


Figure 6. Recognition activity

The red-bounded window includes three buttons aimed to create a new object in different ways as follows (from left to right):

- object of the same type (not necessarily the same combination of properties' trueness), i.e. corresponds to the same path in algoritm of recognition,
- object of the same type (with the same combination of properties' trueness),
- Object of other type.

At a certain stage, students begin to violate the sequence of checking the presence of attributes, starting with an absent feature in conjunctive structures, or from an existing characteristic in disjunctive ones (if there are such features). This means that the recognition algorithm is learned, "curled up" and the use of software should be stopped.

[4] In some cases, due to specifics of the content of the concept's features, it is not always possible to construct an object of a certain type.

[5] It is shown/hidden by pressing image (8) (Figure 5)

1.6. Action of drawing conclusions from the fact of object's belonging or not belonging to the concept. While the recognition of belonging to the concept is a universally recognized mental operation, which possession is necessary for the assimilation of a concept, there is another, equally important ***mental action of drawing conclusions from the fact of object's belonging or not belonging to the concept***.

The definition of a concept can be seen as an *equivalence* of the *term* and the *logical function of its characteristics*.

This equivalence means that:

- ✓ On the basis of the value of the trueness / falseness of the logical function, a conclusion is made about the applicability/inapplicability of the term. - A*ct of concept recognition*.
- ✓ Based on the value of the trueness / falseness of the the term, a conclusion is made about the trueness of the logical function linking the features. - *Act of drawing conclusions from the fact of belonging/not belonging to the concept*.

The act of drawing conclusions, like the act of recognition, is also based on examples and counterexamples, more precisely: on "*imaginary*" *examples* and *counterexamples*. In fact, the specifics of objects is absent here, we are talking about types in pure form.

Figure 7 presents tables of task types in cases of concepts with conjunctive(a), and disjunctive(b) definitions with two properties. Characters *t, ?, f (true, unknown, false)* denote the "*given*" in the task. Sign __ denote features, whose trueness should be discovered. Thus, the following example presents the 6th raw in the table (a):

> - A figure is not a right triangle, although it contains a right angle. *What can you say about it*? (In case of difficulty: *Is it a triangle*?)

It is clear how to formulate similar tasks based on these tables. Microsoft Excel here also can help to make these tasks and the process of their solution more sensible and to "materialize" mental activity, thereby facilitating the process of interiorisation.

By pressing the proper *hat* image one hides (next press - unhide) the feature trueness input buttons (Figure 8). However, all these buttons remain accessible. Pressing them is simulated by typing the corresponding number (the numbers 1-3 correspond to the left triple of buttons, 4-5 – central triple, and 6-9 - the right one).

1	2	$\in$, $\notin$
■	■	t
■	■	f
f	■	f
■	f	f
t	■	f
■	t	f
?	■	f
■	?	f
t	■	?
■	t	?

a)

1	2	$\in$, $\notin$
■	■	t
■	■	f
f	■	t
■	f	t
t	■	t
■	t	t
?	■	t
■	?	t
f	■	?
■	f	?

b)

Figure 7. Drawing conclusions activity tasks' types: a) con*junctive definition, b) disjunctive definition*

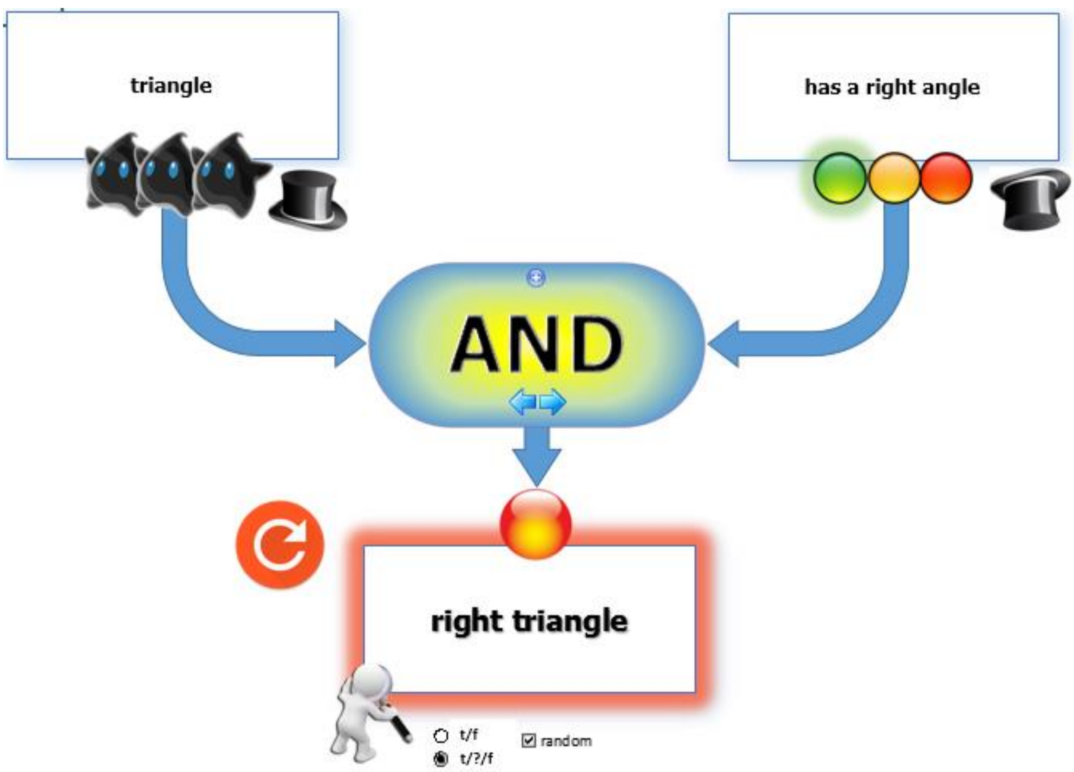


Figure 8. Drawing conclusions

One can sequentially type 1↵, 2↵, 3↵ and emphasize this way the difference of halo color around the term's text box – the trueness of belonging to concept. Remains the question: *button of what color is highlighted in the hidden group?* Secondly, press on the *hat* image to unhide the "secret".

Summarizing, it should be noted that the ***process of finding the proof of theorems or proof tasks and the proof itself consists of chains of actions of recognition and drawing conclusions, based on known definitions of concepts, axioms and theorems***[6] (Figure 9).

The proposed methodology for the study of concepts forms and develops the necessary skills, often long before the students' first encounters with theorems and their proofs thereby removing the stress of novelty, and facilitating understanding.

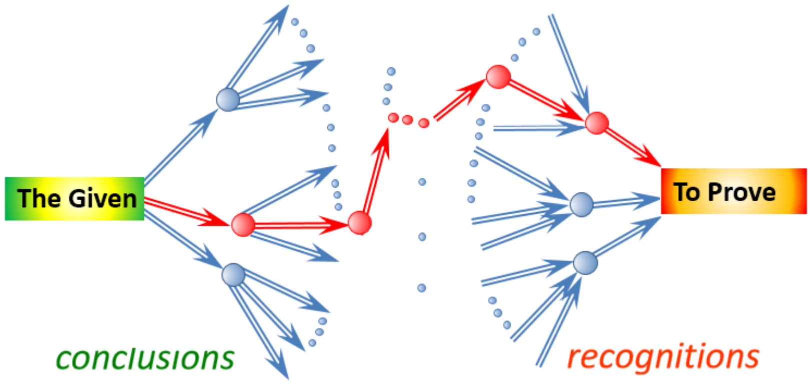


Figure 9. Proof search components: chains of recognition and drawing conclusions activities and the insight path (in red)

Experiment

The experimental verification of the proposed way of studying concepts was carried out with 10-11 years old students of two primary school classes for a period of half a year. In the course of the experiment, it was detected that the students found it possible to separate the logic of definitions from the specific content of their features. Students increasingly used the expressions "first feature", "second feature" instead of "sides are equal", "right angle", etc.

For this reason, the final test included the tasks for completing the truth tables, for example:

has first feature	has second feature	has first feature **AND** has second feature
false	*false*	
false	*true*	
true	*false*	
true	*true*	

Testing the skills of the logical action of drawing conclusions was carried out on tasks with a trivial content (no mathematics), so as to exclude possible mistakes of misunderstanding the subject of the task, for example:

[6] In the famous methodology of G. Polya, related to solving problems, one can find the actions of drawing conclusions in the recommendation to “Separate the various parts of the condition” [5, p.239], and actions of recognition in “starting from the unknown (or the conclusion) and working backwards” [5, p.200].

In the box are wooden and plastic balls of different colors. All the wooden balls are red.
- *Ana took a wooden ball out of the box. Is this ball red?*
- *Mike took a red ball out of the box. Is this a wooden ball?*
- *Tom took out a plastic ball out of the box. Is this a red ball?*
- *Dan took a blue ball out of the box. Is this a wooden ball?*

The grades of the test were surprisingly very high. This prompted us to offer this test to other groups and compare the results. It turned out that the students' assessments of the experimental classes were statistically higher not only in comparison with their peers not participating in the experiment, but even in comparison with the results of the high school graduates.

Conclusions

In result of the study:

1. *Was proved the **importance** and showed the **function** and **place** of **counterexamples** in mathematics studies.*
2. *Was confirmed the fundamental role of the mental actions of **recognition**, and of **drawing conclusions from the fact of belonging or not belonging to the concept** in the process of mastering the concept as well as **in the development of students logical thinking, in formation of students' abilities and needs in proof** .*
3. *Was created an accurate and **optimal typology of relevant tasks**.*
4. *Was provided a **general algorithm of construction of such systems of types**.*
5. *Experimental results show the **effectiveness of the proposed approach**, **supported by technological aids**.*

Supplementary Electronic Materials

Videos with animations: https://sites.google.com/view/counterexamples-in-math-edu-1/

References

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